Extreme vorticity growth in Navier-Stokes turbulence

Measuring intense rotation and dissipation in turbulent flows

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Extreme vorticity growth in Navier-Stokes *solutions*

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Limits on Enstrophy Growth for Solutions of the Three-dimensional Navier-Stokes Equations

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Navier-Stokes equations:

$$\dot{\vec{u}} + \vec{u} \cdot \vec{\nabla}\vec{u} + \vec{\nabla}p = \nu \Delta \vec{u}$$

$$0 = \vec{\nabla} \cdot \vec{u}$$

- Periodic box: $\vec{x} \in [0,L]^3$
- Initial condition: $\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x})$

$$\left(\text{WLOG } \int \vec{u}_0(\vec{x}) d^3x = 0\right)$$

***Open* (\$1M) *question*:**

Does a unique smooth solution exist for all $t > 0$?

Some definitions & some things we know:

- Kinetic energy:

$$K(t) \equiv \tfrac{1}{2}\int \left|\vec{u}(\vec{x},t)\right|^2 d^3x = \tfrac{1}{2}\left\|\vec{u}(\vec{x},t)\right\|_2^2$$

- Vorticity:

$$\vec{\omega} = \nabla \times \vec{u} \quad \Rightarrow \quad \dot{\vec{\omega}} + \vec{u}\cdot\vec{\nabla}\vec{\omega} = \nu\Delta\vec{\omega} + \vec{\omega}\cdot\vec{\nabla}\vec{u}$$

- Enstrophy:

$$E(t) \equiv \left\|\vec{\omega}(\vec{x},t)\right\|_2^2 = \left\|\vec{\nabla}\vec{u}(\vec{x},t)\right\|_2^2 \geq \frac{8\pi^2}{L^2}K(t)$$

- If solution is *smooth* enough, $dK/dt = -\nu E$.

Global (in time) ***weak*** solutions exist:

- If $\boldsymbol{K_0}<\infty$, there are weak solutions with finite energy,

$$K(t) \le K_0 \quad \forall t \ge 0$$

- … and with finite ***integrated enstrophy***,

$$\int_{t_a}^{t_b} E(t)dt \ < \ \infty \quad \forall 0 \le t_a \le t_b$$

- … but only known to satisfy an energy ***inequality***,

$$K(t_b) \le K(t_a) - \nu \int_{t_a}^{t_b} E(t)dt \quad \text{for a.e. } t_a > 0$$

- … and there is ***no*** assurance that they are unique.

Local (in time) ***strong*** solutions exist:

- For $(8\pi^2/L^2)K_0 \le \boldsymbol{E_0} < \infty$,

$$\exists\, T(K_0, E_0, \nu) > 0 \ni E(t) < \infty \text{ for } 0 \le t < T.$$

- **Fact:**

$$E(t) < \infty \text{ for } t_a \le t \le t_b$$
$$\Updownarrow$$
$$\vec{u}(\cdot,t) \in C^\infty([0,L]^3) \text{ for } t_a < t \le t_b.$$

- And strong solutions are ***unique***.

As long as the enstrophy is finite ...
or for Galerkin-regularized solutions

$$\frac{dK}{dt} = -\nu E$$

$$\frac{dE}{dt} = -2\nu \left\| \vec{\nabla} \vec{\omega} \right\|_2^2 + 2 \int \vec{\omega} \cdot \vec{\nabla} \vec{u} \cdot \vec{\omega} \, d^3 x$$

$$= \underbrace{-2\nu \left\| \Delta \vec{u} \right\|_2^2 + 2 \int \vec{u} \cdot \vec{\nabla} \vec{u} \cdot \Delta \vec{u} \, d^3 x}$$

$$\Uparrow$$

Enstrophy generation rate $G\{\boldsymbol{u}\}$ = ***production*** – ***dissipation***

Vortex stretching & enstrophy production:

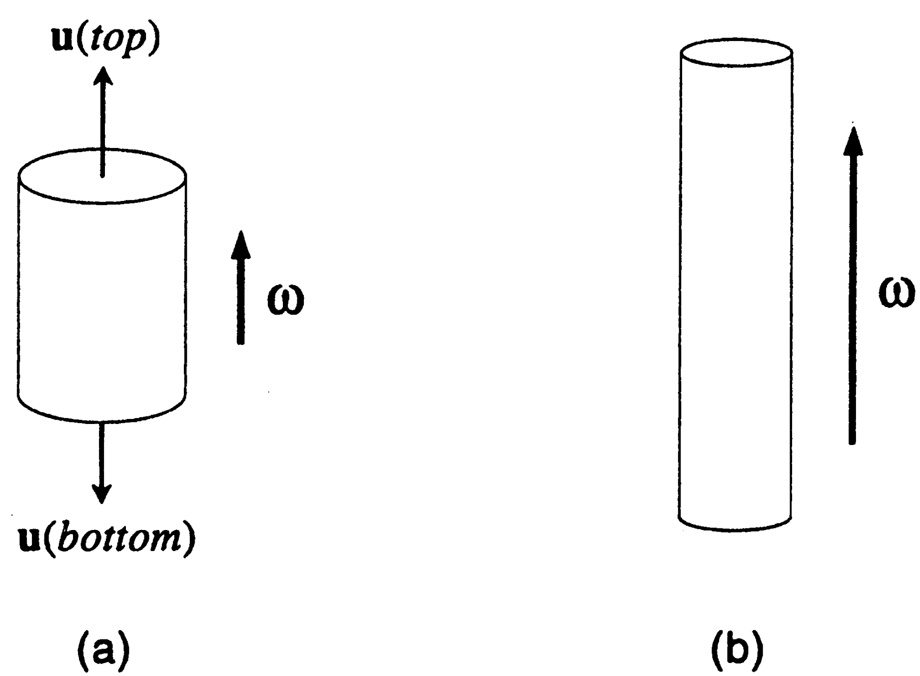


Fig. 1.4. The vortex stretching mechanism. When $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ has a component parallel to $\boldsymbol{\omega}$, as in (*a*), the fluid element is stretched in the direction of the vorticity. The resulting decrease in the element's moment of inertia, illustrated in (*b*), leads to an increase in the amplitude of the vorticity.

Vorticity can be amplified; enstrophy can be produced.

Does this nonlinear process get out of control?

System of differential (in)equations:

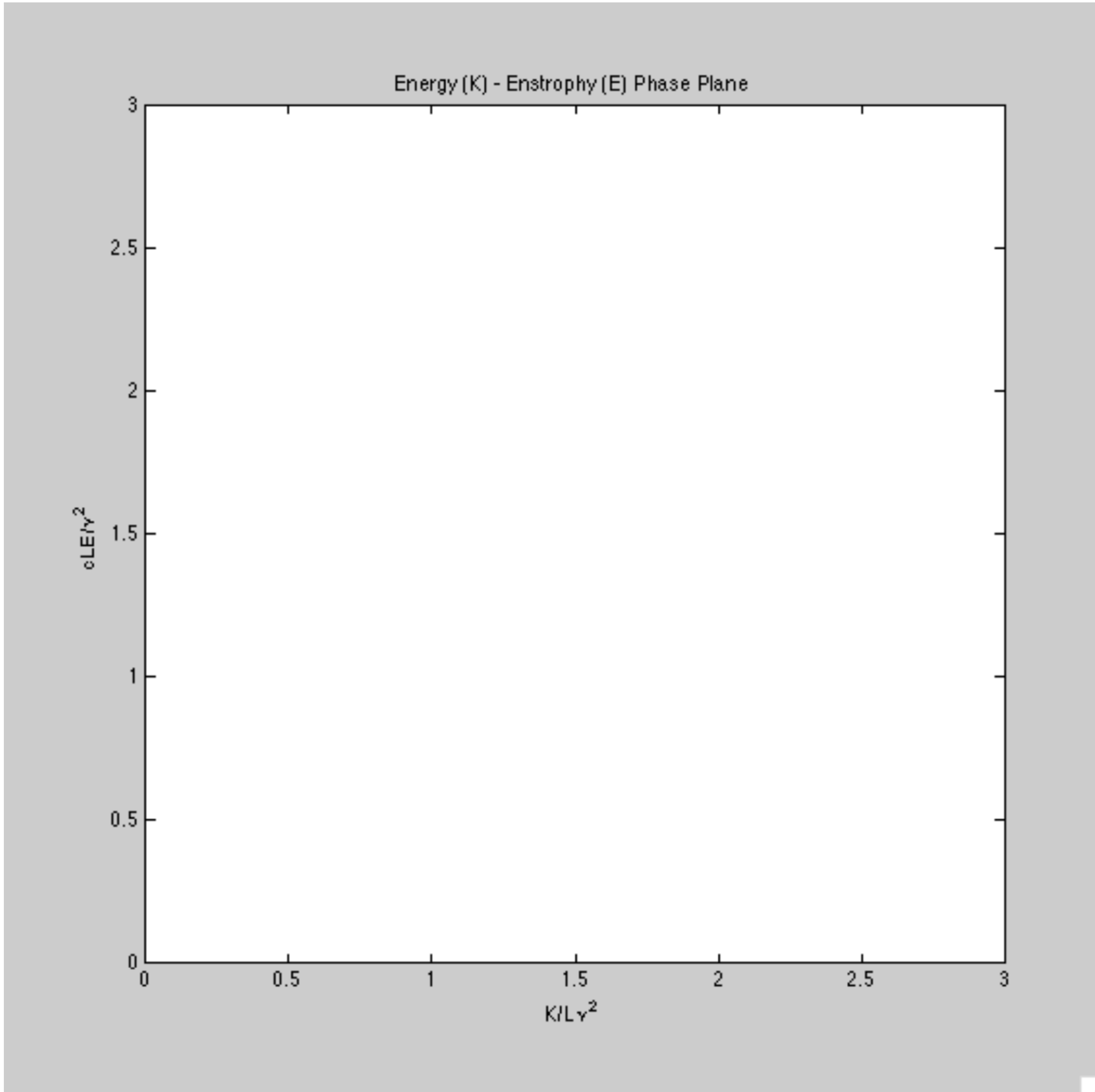
$$\frac{dK}{dt} = -\nu E$$

$$\frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

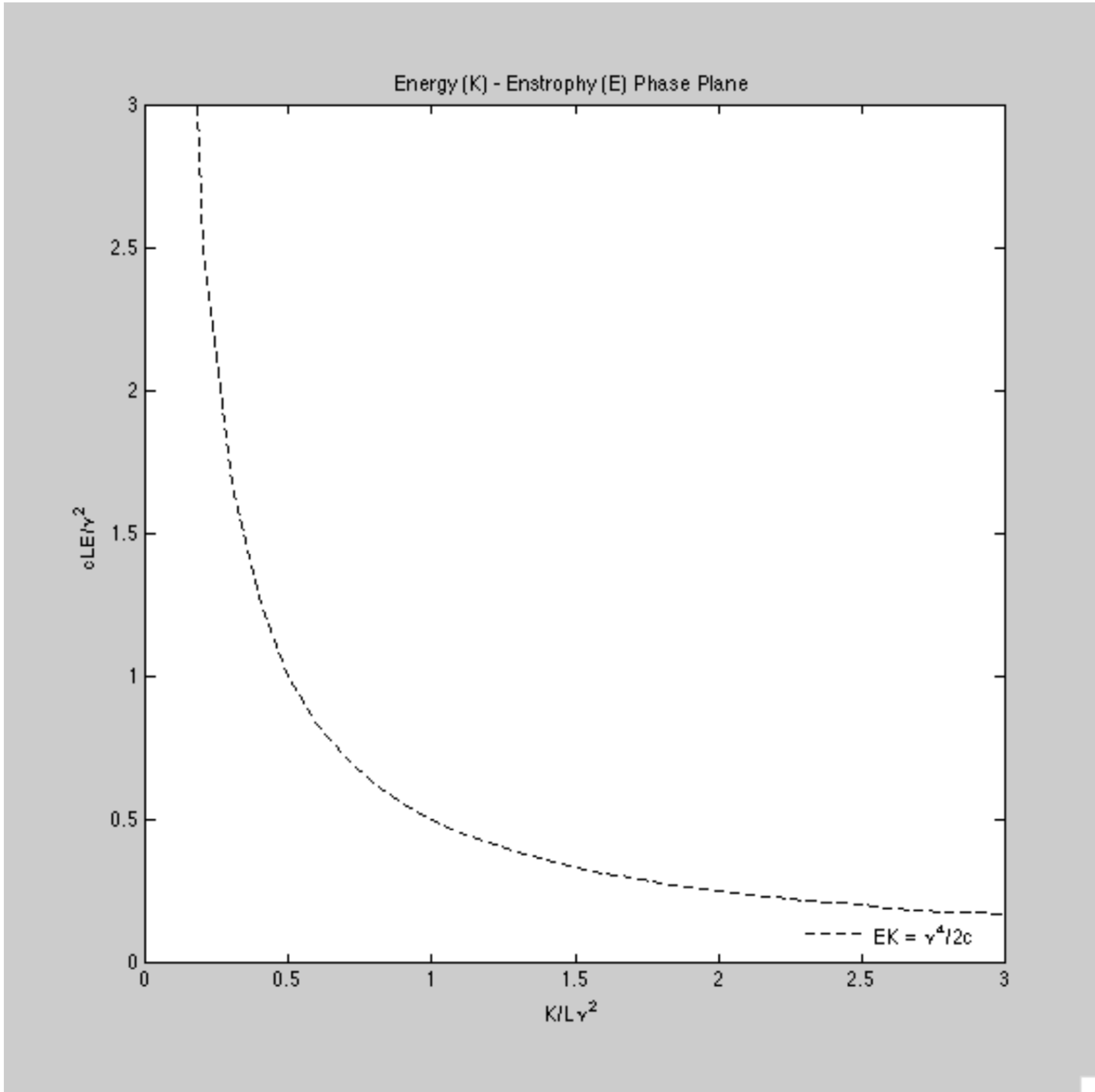
$$\Rightarrow \quad E \le E_0\left(\frac{K}{K_0}\right)^{\frac{1}{2}}\left(1 + \frac{2cK_0E_0}{\nu^4}\left[\left(\frac{K}{K_0}\right)^{\frac{3}{2}} - 1\right]\right)^{-1}$$

... as long as RHS $\ge$ 0.

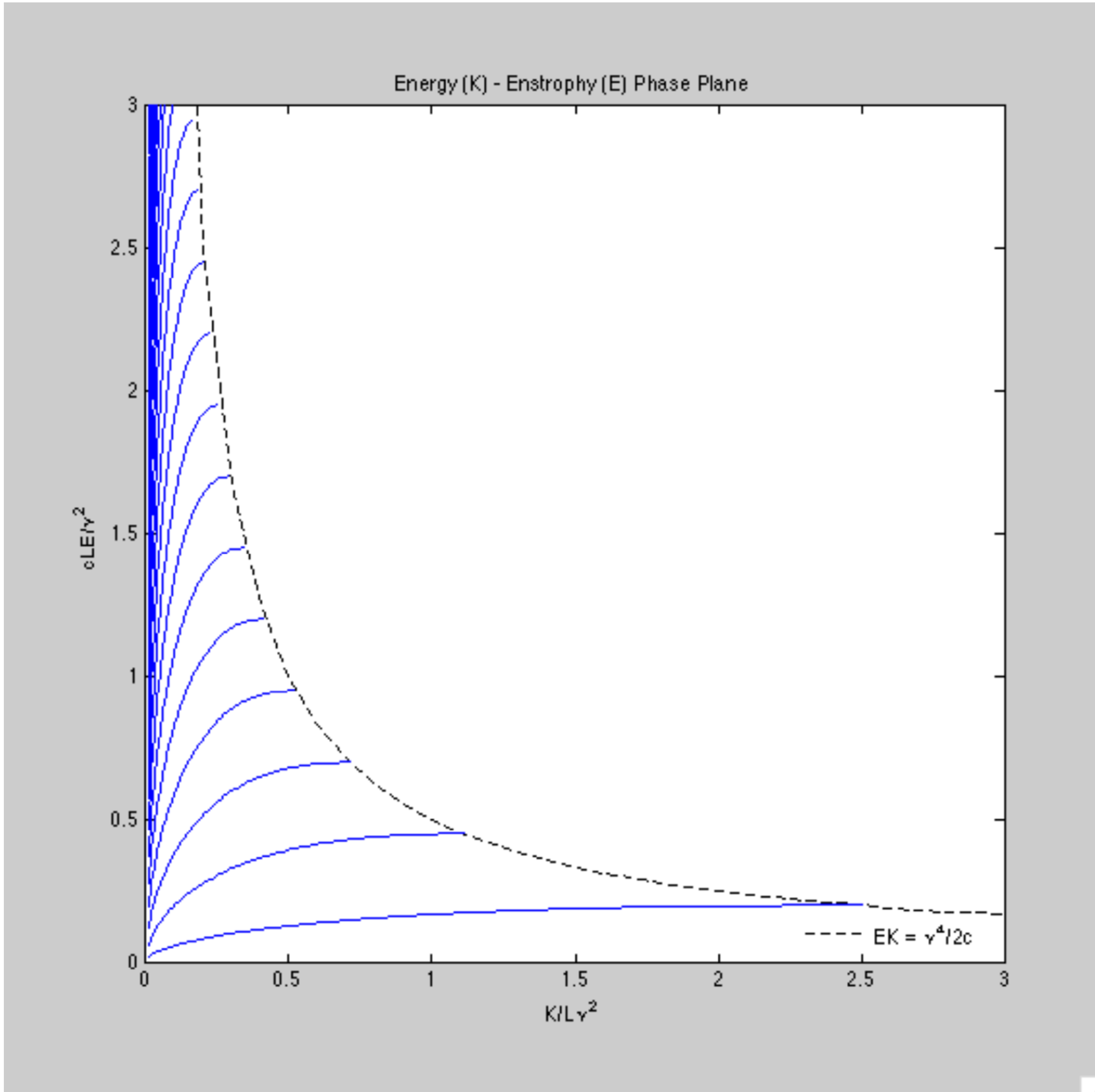
$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

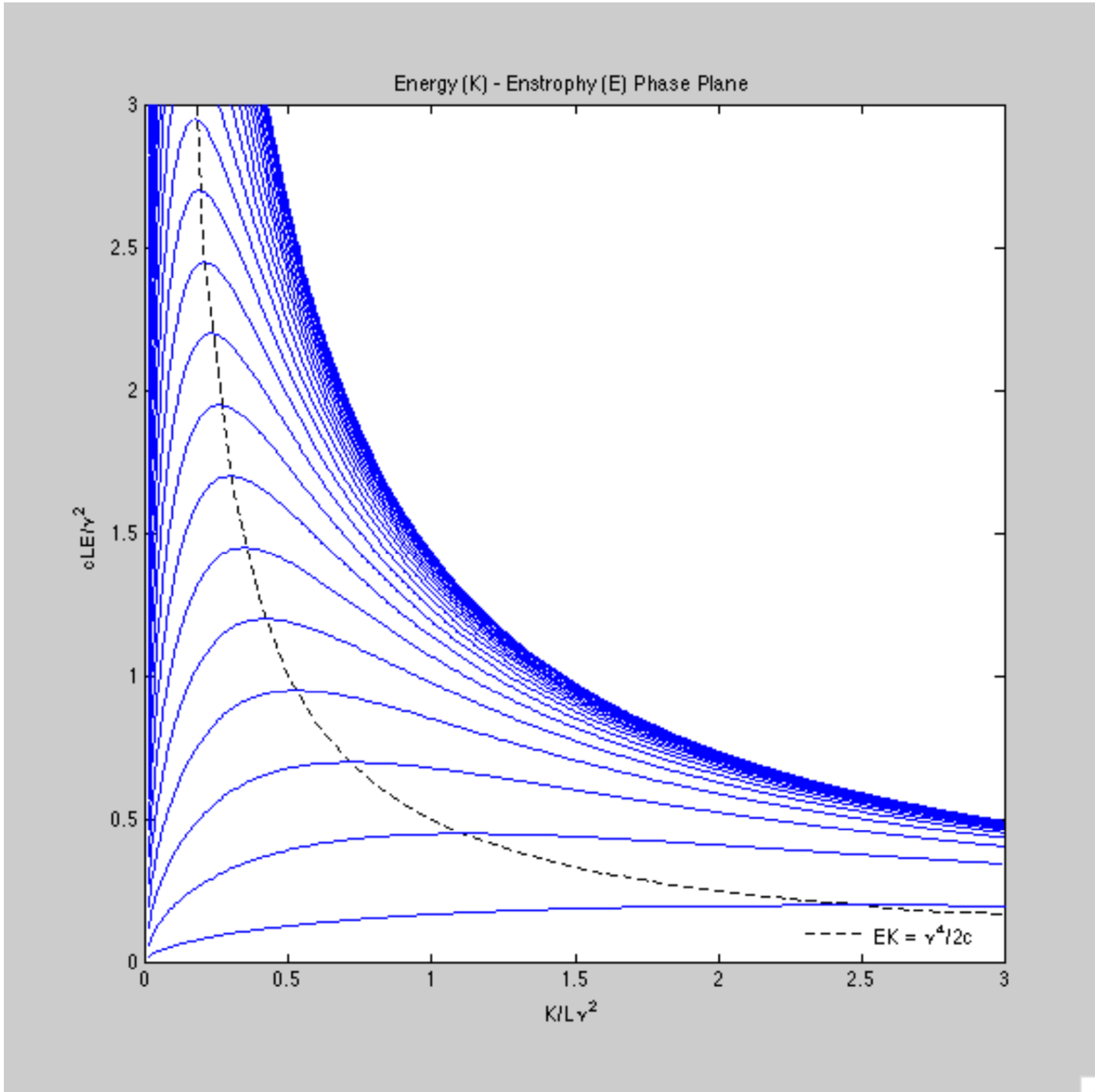


$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

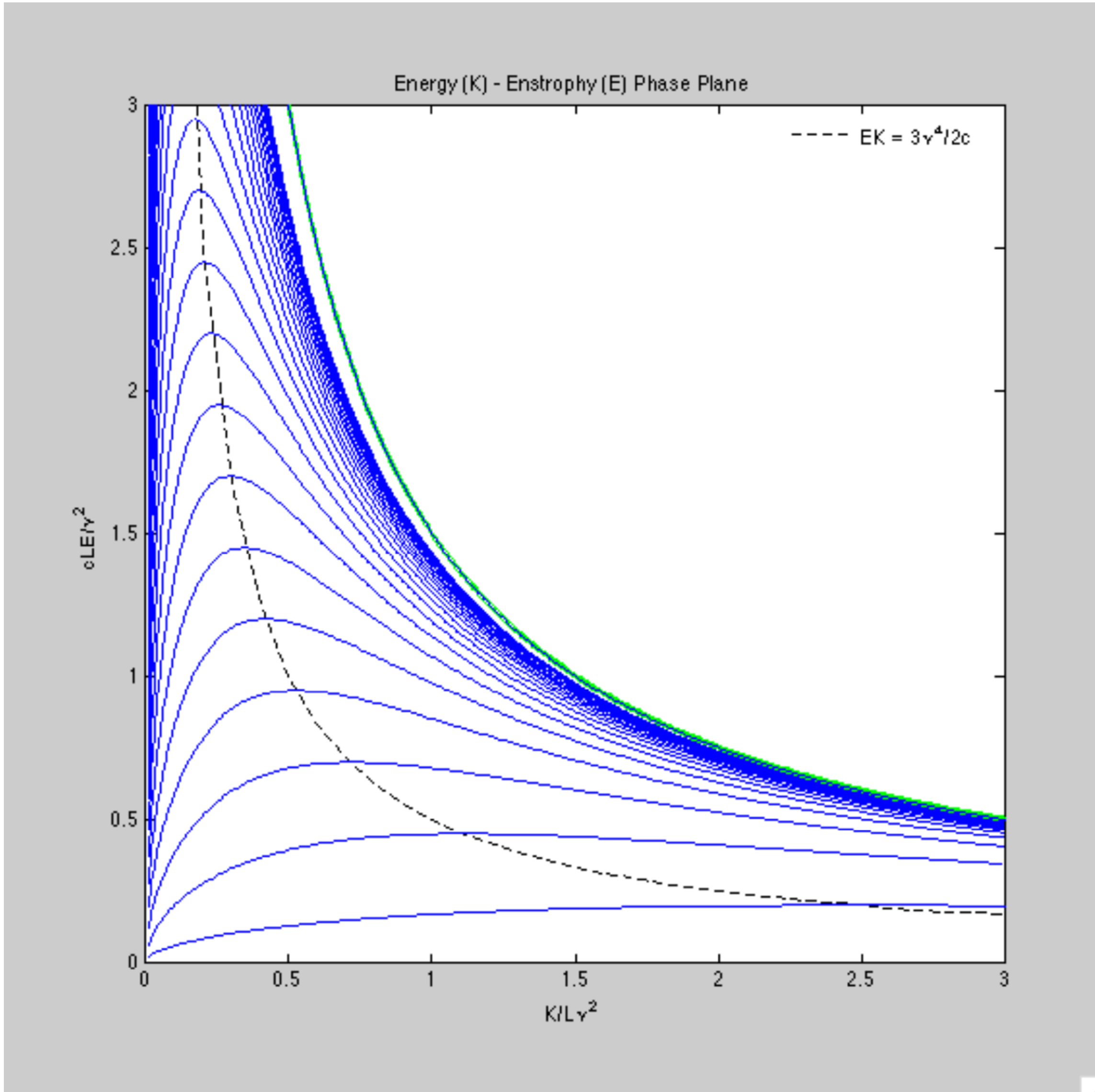


Enstrophy ***decreases*** if $E_0 K_0 \le \nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

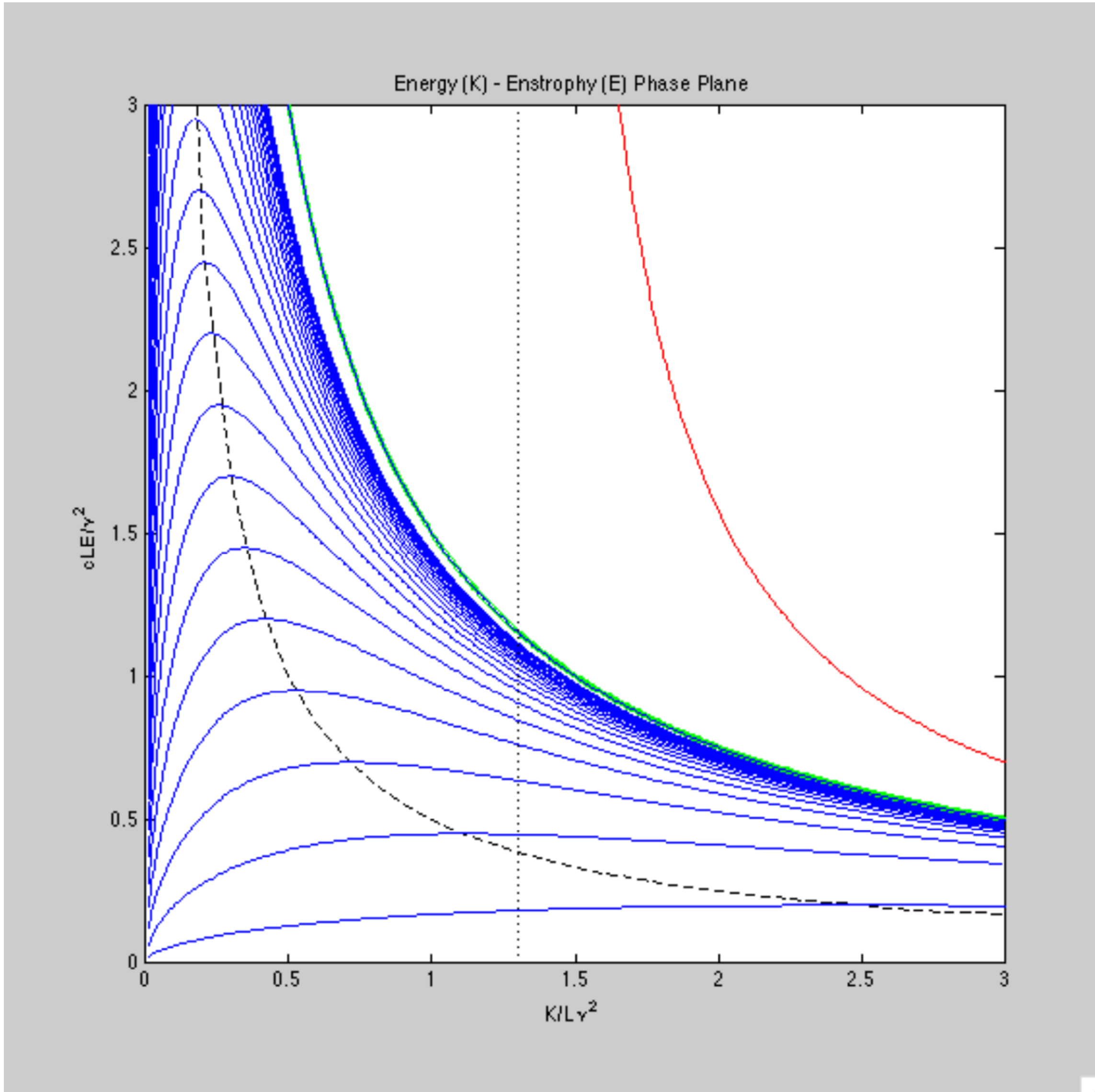


$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



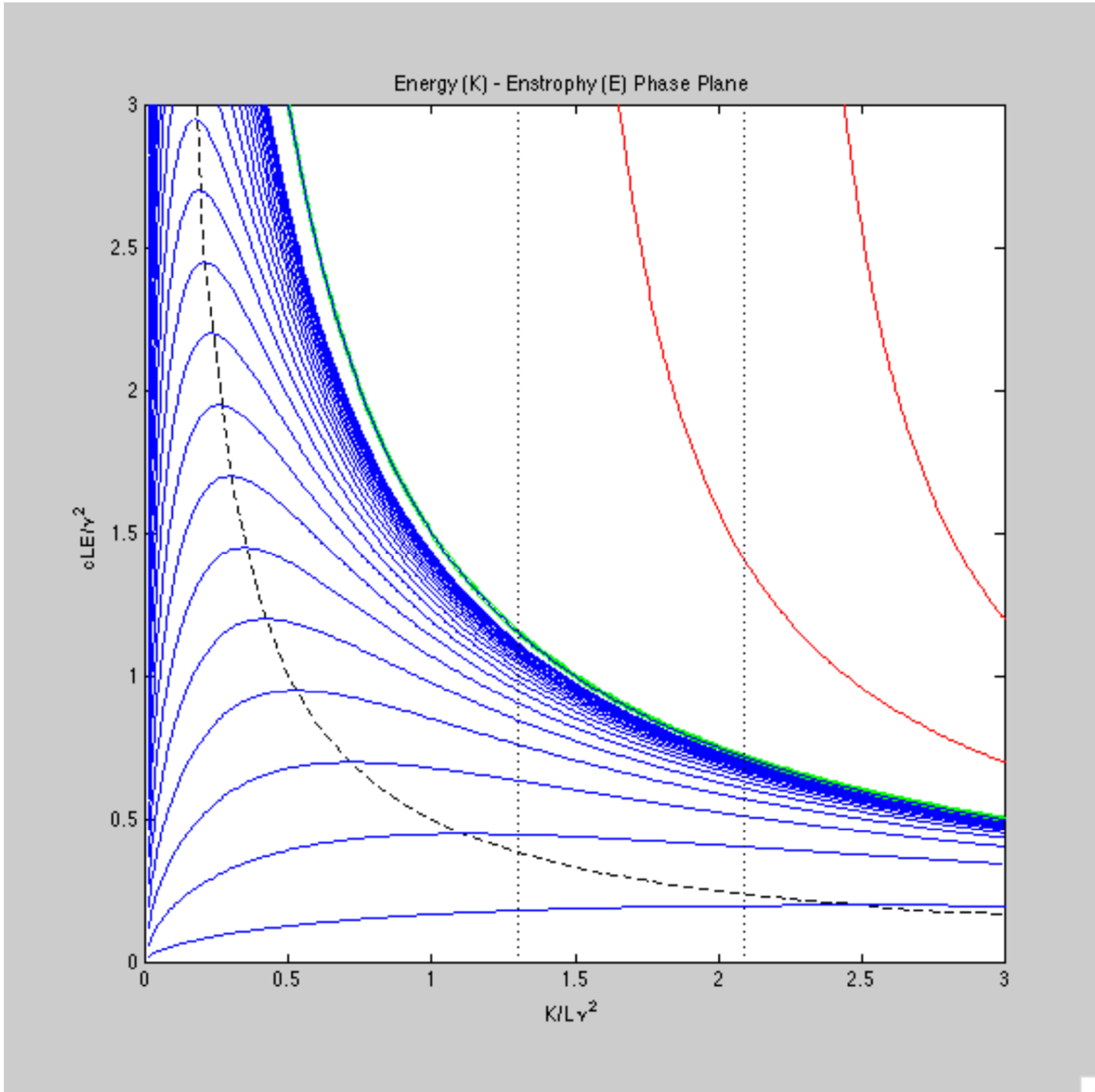
Global existence and uniqueness ***if*** $E_0K_0 \le 3\nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



But does ***not*** prevent finite-time singularity if $E_0 K_0 > 3\nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \leq -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



But does ***not*** prevent finite-time singularity if $E_0 K_0 \geq 3\nu^4/2c$.

Question:

How big can $G\{\boldsymbol{u}\}$ ***really*** get in terms of K and E?

- Analytic estimates ***don't*** account for div $\boldsymbol{u} = 0$ …
- or ***total*** competition between production & dissipation.
- Would like to solve the variational problem for max rate:

$$M(K,E) = \sup_{\vec{\nabla}\cdot\vec{u}=0} \left\{ G\{\vec{u}\} \mid \tfrac{1}{2}\|\vec{u}\|_2^2 = K \text{ and } \left\|\vec{\nabla}\vec{u}\right\|_2^2 = E \right\}$$

Settle for slightly less:

$$\Re(E) = \sup_{\vec{\nabla}\cdot\vec{u}=0}\left\{G\{\vec{u}\} \;|\; \left\|\vec{\nabla}\vec{u}\right\|_2^2 = E\right\}$$

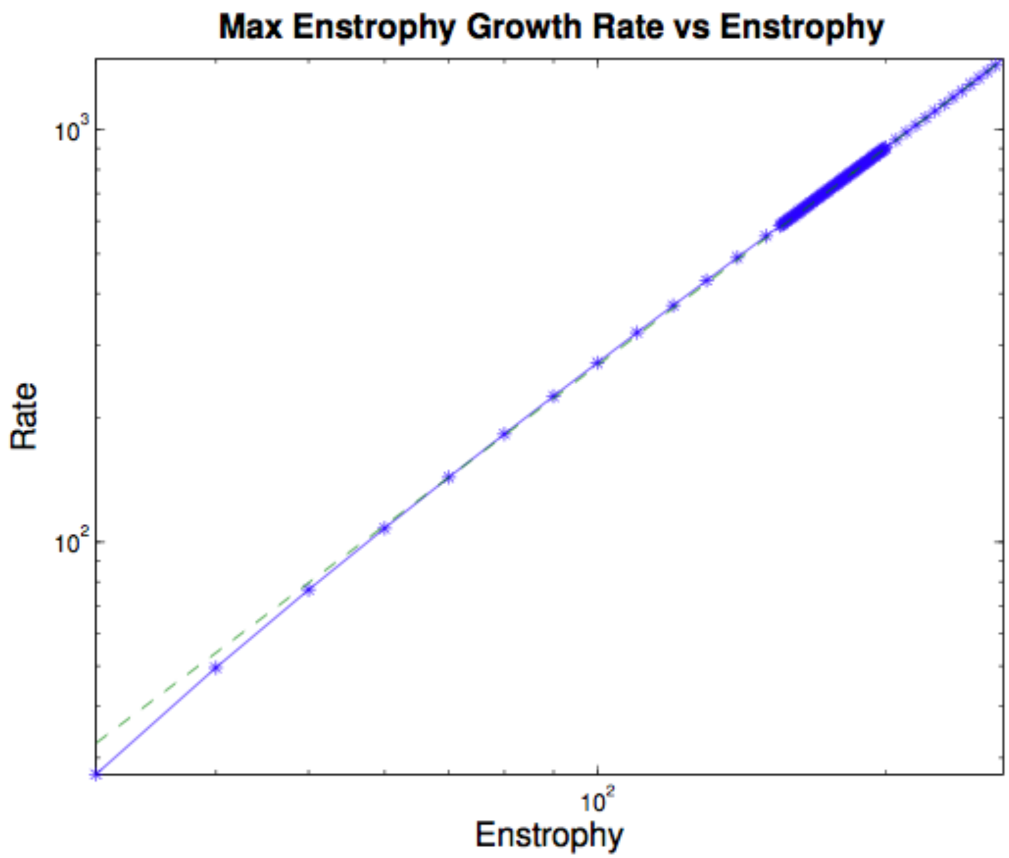
$$\text{so} \quad \frac{dE}{dt} \le \Re(E)$$

- We know that $\Re \le cE^3/\nu^3$ … but that $\neq$ **$1M**.
- "*Critical*" behavior is $\Re \sim E^2$ as $E \to \infty$.
- Solve the Euler-Lagrange equations:

$$0 = \frac{\delta}{\delta\vec{u}}\left\{G\{\vec{u}\} + \int p\vec{\nabla}\cdot\vec{u}\,d^3x + \lambda\int\left|\vec{\nabla}\vec{u}\right|^2 d^3x\right\}$$

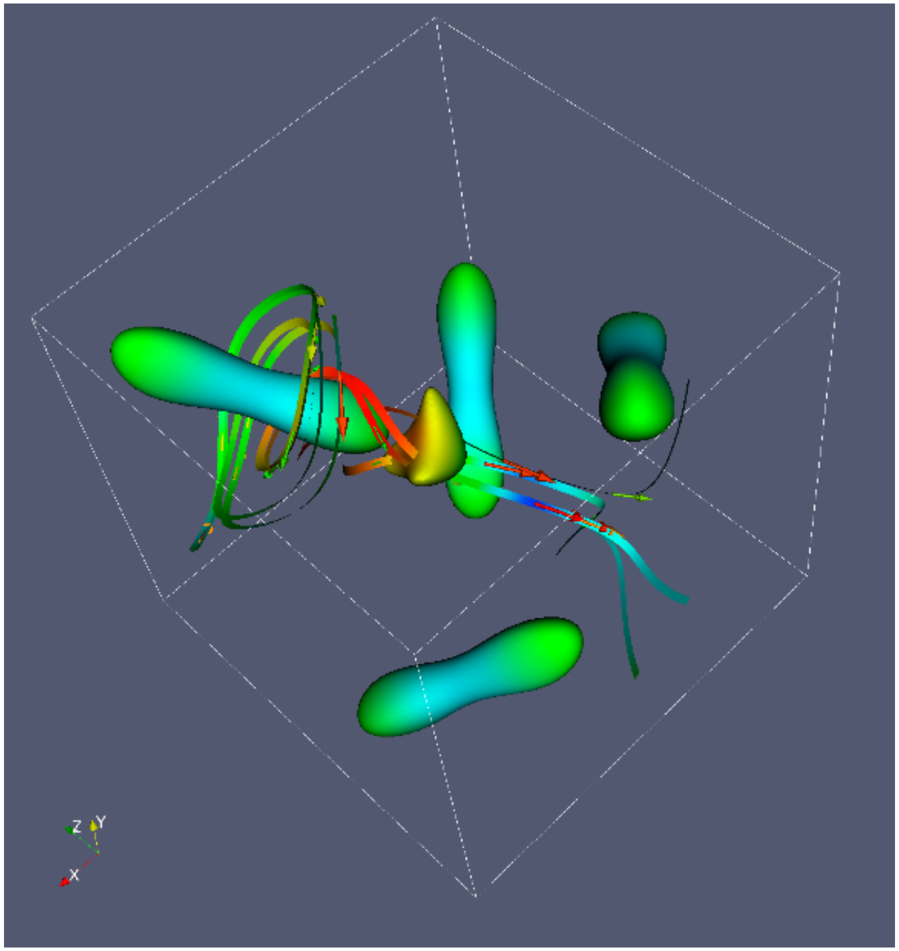
Solve computationally … via ***gradient ascent method***.

Starting from exact solution as $E \to 0$ ...



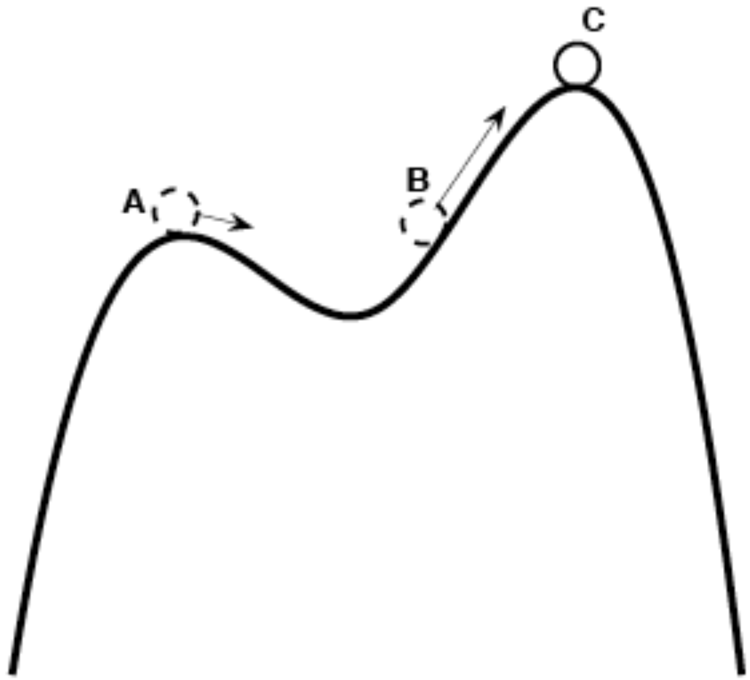
- Large E behavior is $\Re \sim E^{1.78\,(=7/4?)}$ ... ***subcritial!***

What do the maximizers look like?



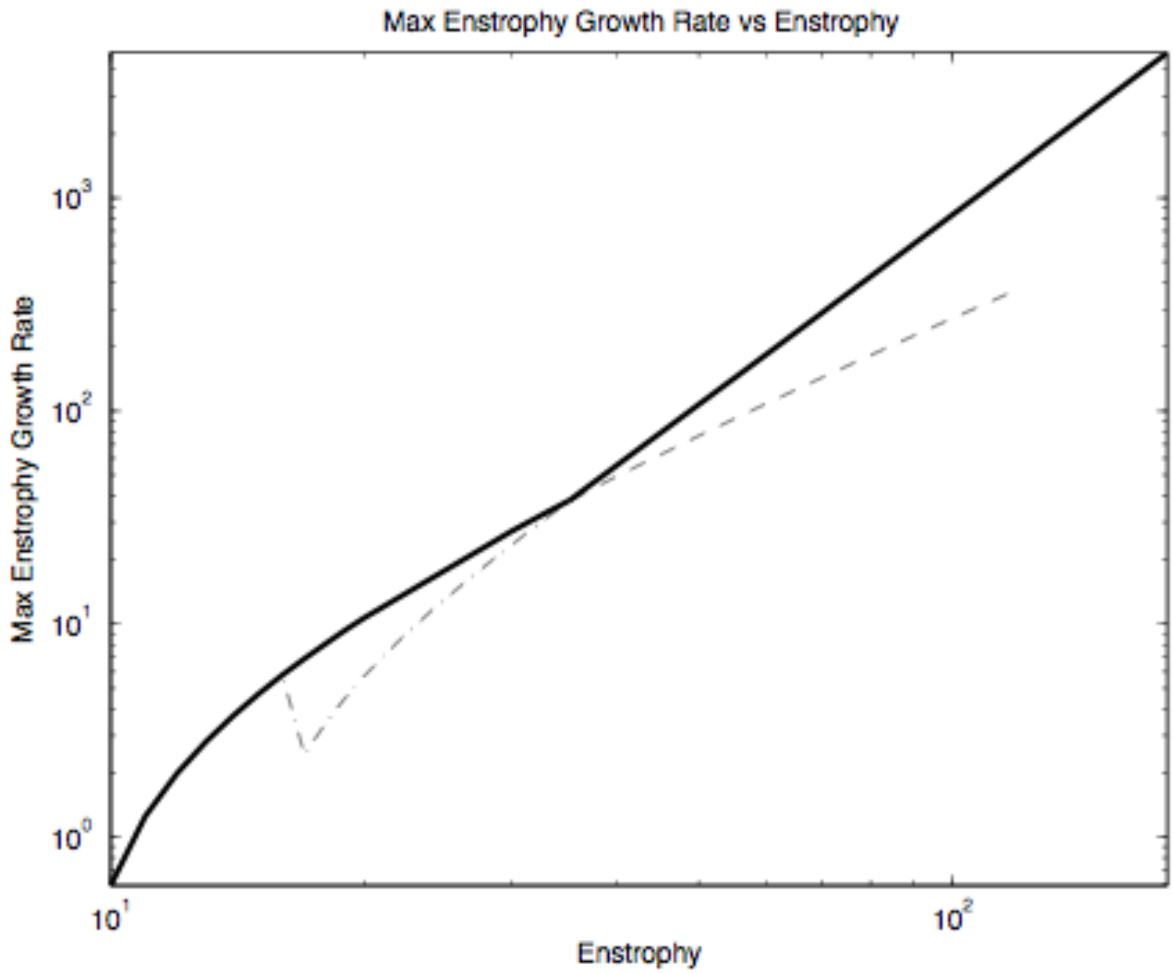
periodic array of "vortex stretchers"

But this is a non-convex variational problem



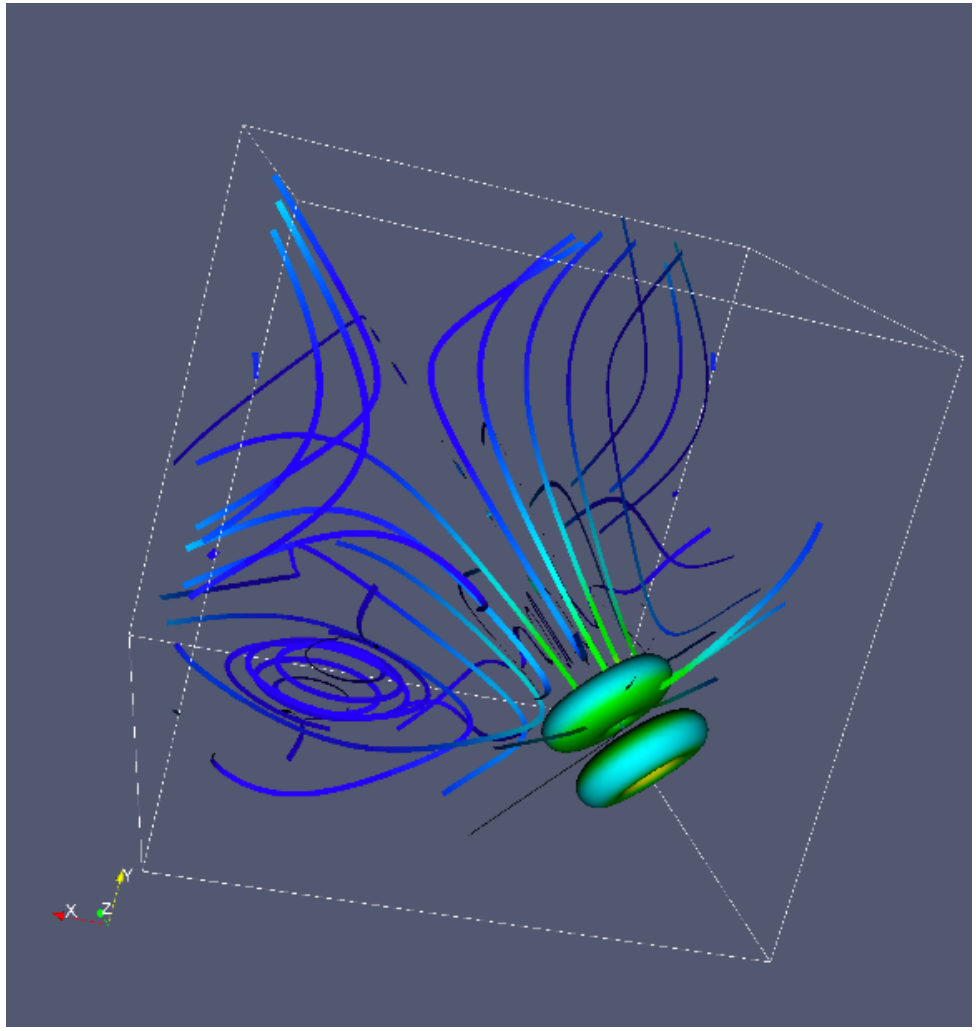
- Euler-Lagrange solutions are local extrema …
- So must see if there are other, global, maxima.

… another branch emerges at high *E*:



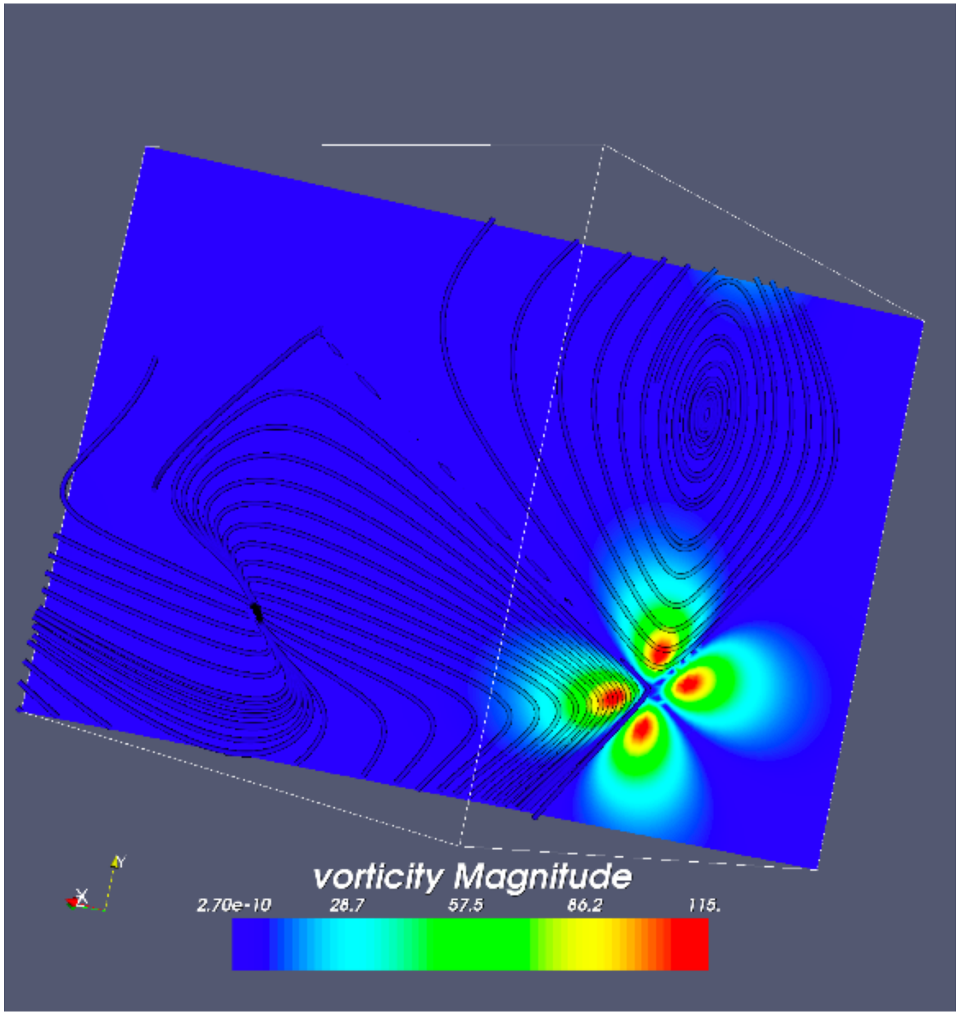
- Large E behavior is $\Re \sim E^{2.997\ (= 3?)}$ … ***as estimated***.

What do these maximizers look like?



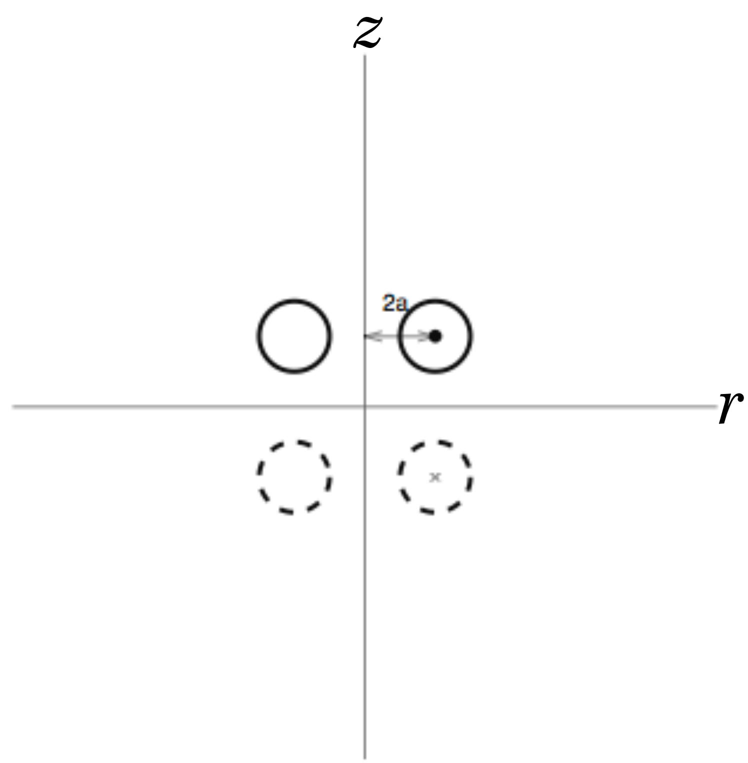
colliding vortex rings

Another view ...



Vorticity in a plane slice

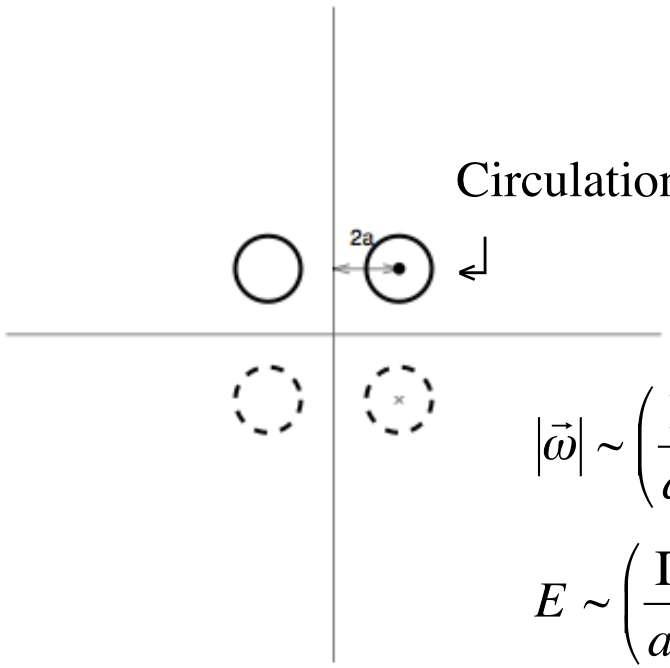
Reality check:



For velocity fields with cylindrical symmetry …

$$G\{\vec{u}\} = -2\nu \left\| \vec{\nabla}\vec{\omega} \right\|_2^2 + 2\int \omega_\theta^2 \frac{u_r}{r}\; d^3x$$

Reality check (continued):



$$G\{\vec{u}\} = -\nu \left\| \vec{\nabla}\vec{\omega} \right\|_2^2 + 2\int \omega_\theta^2 \frac{u_r}{r}\, d^3x$$

$$\sim -\nu \left(\frac{1}{a}\frac{\Gamma}{a^2} \right)^2 a^3 + \left(\frac{\Gamma}{a^2} \right)^2 \left(\frac{1}{a}\frac{\Gamma}{a} \right) a^3$$

$$|\vec{\omega}| \sim \left(\frac{\Gamma}{a^2} \right)$$

$$E \sim \left(\frac{\Gamma}{a^2} \right)^2 a^3 = \frac{\Gamma^2}{a}$$

$$\sim -\nu \frac{E}{a^2} + \frac{E^{3/2}}{a^{3/2}}$$

← **Remember this**

Then maximize over a …

… max occurs at $a \sim \nu^2/E$

$$\Rightarrow \quad G\{\vec{u}\} \sim \frac{E^3}{\nu^3}$$

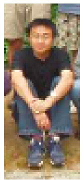
Remarks & laments:

- **Remark #1**: The analytic asymptotic high-E estimate $\Re(E) \leq cE^3/\nu^3$ ***can*** be saturated by divergence-free fields.
- **Lament #1**: no **$1M** to be found down this road!

- **Remark #2**: This "most dangerous" velocity field will ***not*** produce a singularity in N-S.
- **Lament #2**: no **$1M** to be found down that road!

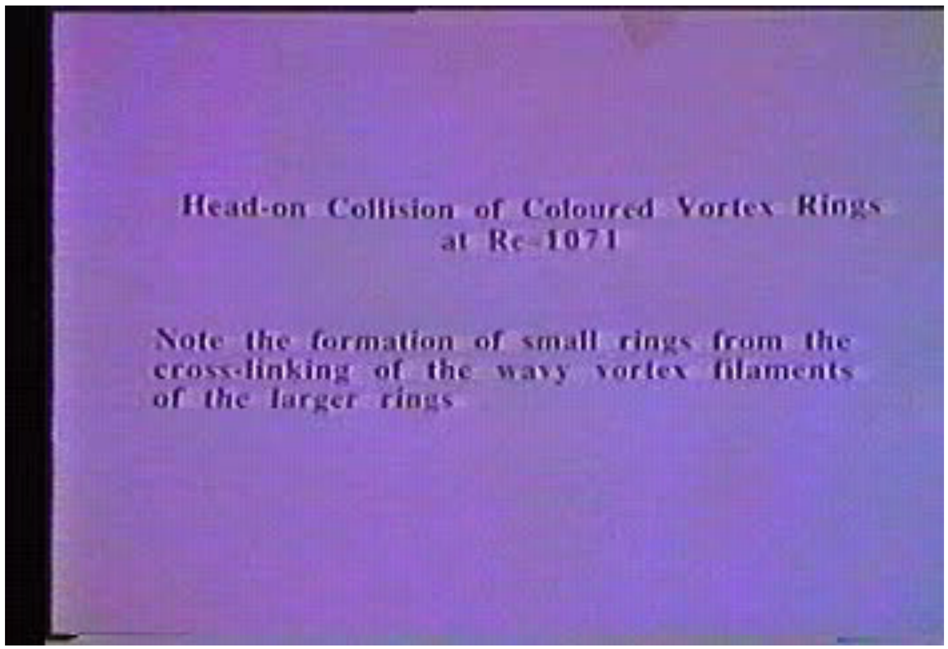
- **Remark #3**: $K \sim 1/E$ for the optimizer, so we're not sure if knowing the full upper limit ***M(K,E)*** will help …
- **Lament #3**: so **$1M** not ***clearly*** down that road, either!

Maybe Lu will find **$1M** in Manhattan ...



Just for fun … what *do* colliding vortex rings do?

(from website of Dr. T.B. Nickels <http://www2.eng.cam.ac.uk/~tbn22/Mov.html>)



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Extreme vorticity growth in Navier–Stokes turbulence

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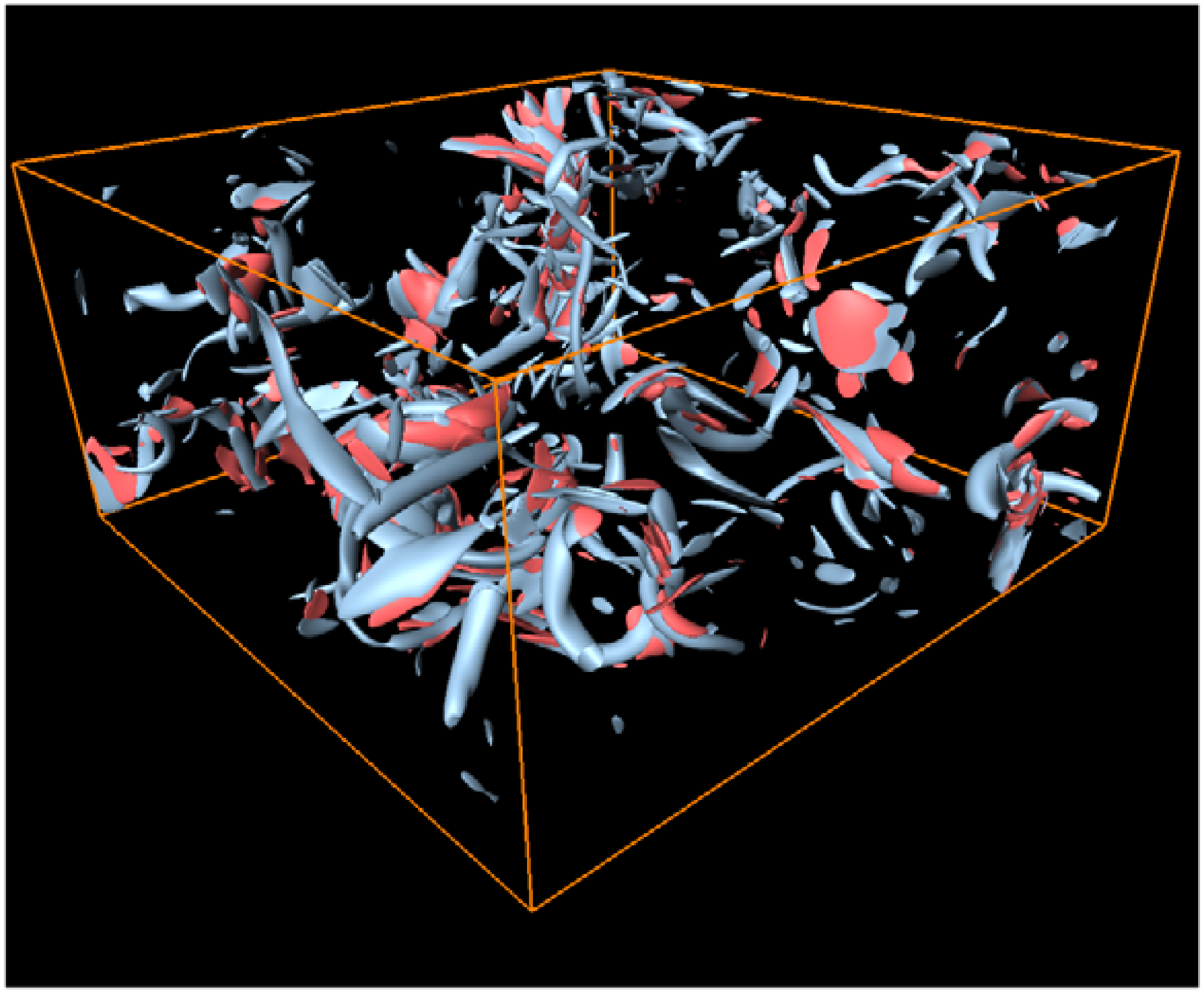


Fig. 1. (Color online.) Vortex tubes and dissipation sheets in homogeneous isotropic turbulence. Isosurfaces of the vorticity magnitude square (or local enstrophy) $\Omega = \omega^2$ (cyan) and the energy dissipation rate $\epsilon = 2\nu S_{ij} S_{ij}$ (red) with $S_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ the rate of strain tensor. Both surfaces are shown at the level of ten times their mean. The displayed volume is $1/16$ of the full simulation box.

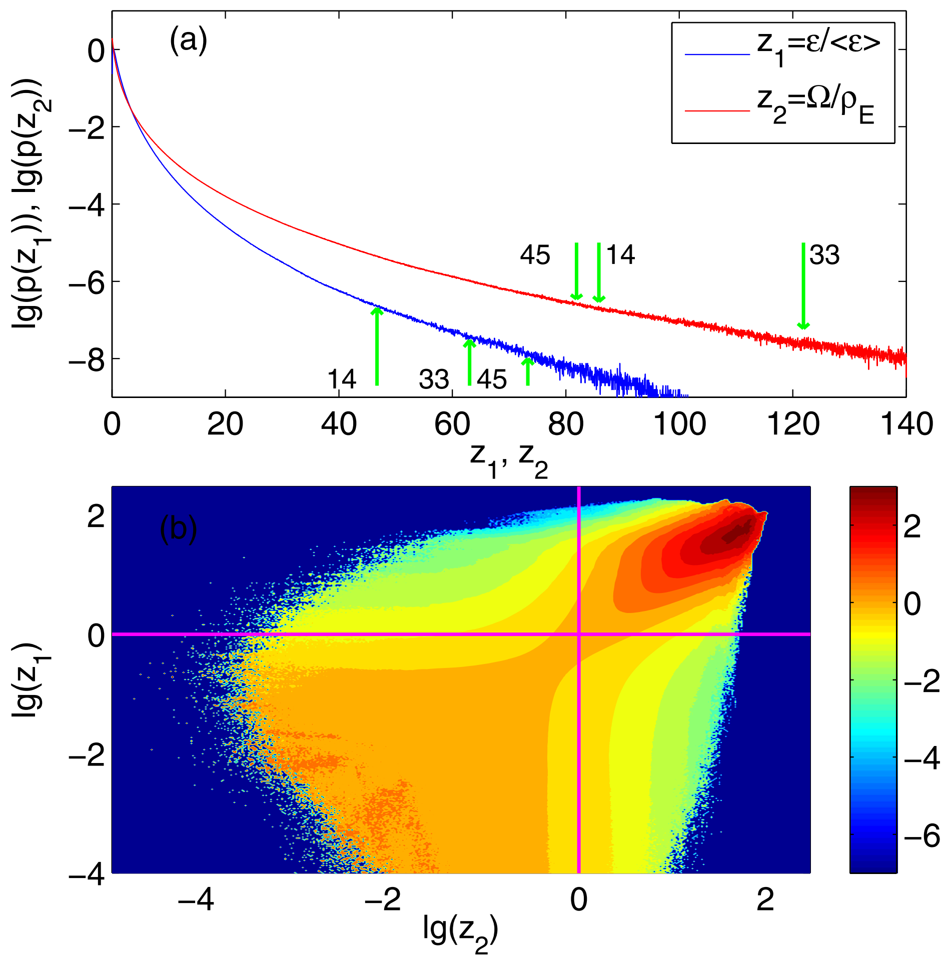


Fig. 2. (Color online.) Statistics of local enstrophy and energy dissipation rate. (a) Probability density functions of local enstrophy, Ω, and energy dissipation rate, ϵ, given in units of their means, respectively. The vertical arrows mark the global maxima of Ω and ϵ, respectively, in case of three Lagrangian tracers, No. 14, 33 and 45. (b) Joint probability density function of local enstrophy and energy dissipation rate. The distribution is normalized by both single quantity distributions, $p(z_1, z_2)/[p(z_1)p(z_2)]$, in order to highlight the statistical correlations between $z_1 = \epsilon/\langle\epsilon\rangle$ and $z_2 = \Omega/\rho_E$. Color coding is in decadic logarithm.

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leqslant \frac{27c^3}{16\nu^3} E(t)^3, \tag{5}$$

There is a second analytical result that pertains to the growth rate of the ensemble averaged squared vorticity (or enstrophy density)

$$\rho_E = \langle \boldsymbol{\omega}^2 \rangle \tag{6}$$

for the particular case of homogeneous and isotropic (box) turbulence. A direct consequence of the von Kármán–Howarth (KH) equation [18] for the velocity correlations, when the volume average $\langle\cdot\rangle$ in (7) agrees with the ensemble average that appears in the KH equation, is derived in [19,20] and states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_E = -\frac{7S}{3\sqrt{15}}\rho_E^{3/2} - 70\nu\left\langle\left(\partial_x^2 u_x\right)^2\right\rangle, \tag{7}$$

where S is the skewness of the longitudinal velocity derivative, and u_x is the x-component of the turbulent velocity field. It is an empirical fact that $S < 0$. It has been observed that the skewness S is basically constant for Taylor microscale Reynolds numbers $R_\lambda \lesssim 200$ and it grows weakly as $|S| \sim R_\lambda^{0.11}$ for $R_\lambda > 200$ [21,22]. Thus we will assume for purposes of discussion and data analysis that $\mathrm{d}\rho_E/\mathrm{d}t \sim \rho_E^{3/2}$ holds approximately. This exponent is much smaller than the one in the upper bound (5).

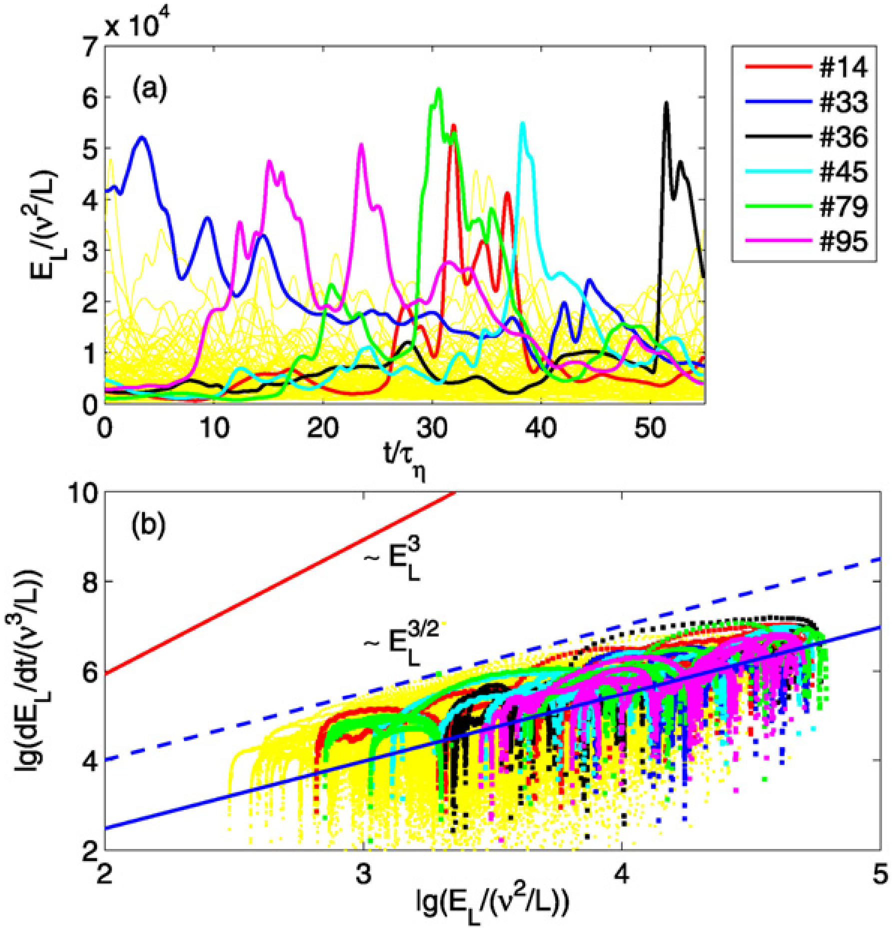


Fig. 3. (Color online.) Quasi-Lagrangian analysis of enstrophy. (a) Time series of $E_L(t)$ for all 100 subvolumes V_L are plotted. The time traces that reach the largest local maxima for E_L are colored differently and their labels are indicated in the legend. Enstrophy is given in units of ν^2/L with $L = 16\eta_K$. (b) Enstrophy growth rate versus enstrophy. The enstrophy growth rate, $\mathrm{d}E_L/\mathrm{d}t$, is given in units of ν^3/L^3. The *a priori* upper bound $\mathrm{d}E_L/\mathrm{d}t = 27\sqrt{2}/(8\nu^3\sqrt{\pi^3})E_L^3$ is indicated as a red line. The growth that follows from the von Kármán–Howarth equation [18], $\mathrm{d}E_L/\mathrm{d}t \approx -7S/3\sqrt{15}E_L^{3/2}$ with a derivative skewness of $S = -0.5$, is indicated as a solid blue line. The dashed blue line has the same slope and serves as a guide to the eye. Color coding is as in panel (a).

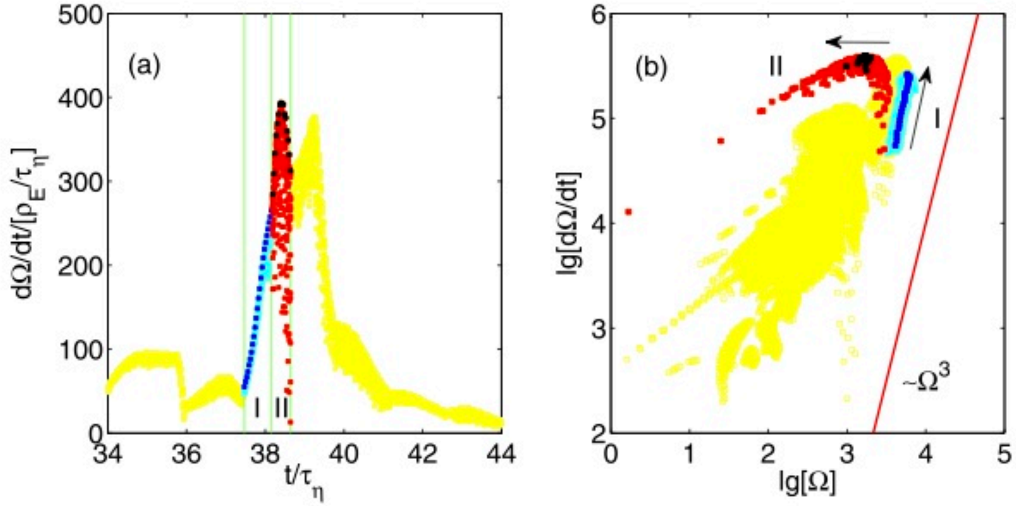


Fig. 4. (Color online.) Local analysis of very rapid enstrophy growth events. (a) Time evolution of the maximum of $d\Omega/dt$ in V_L for tracer No. 45. Growth rates at the grid point of the maximum (blue for I and black for II) and the 27 neighboring points (cyan for I and red for II) are shown. The curve is piecewise continuous since one local maximum in V_L takes over a former at a different grid point. (b) Replot of the data from (a) in the $d\Omega/dt$–Ω plane. The arrows in panel (b) indicate the time evolution.

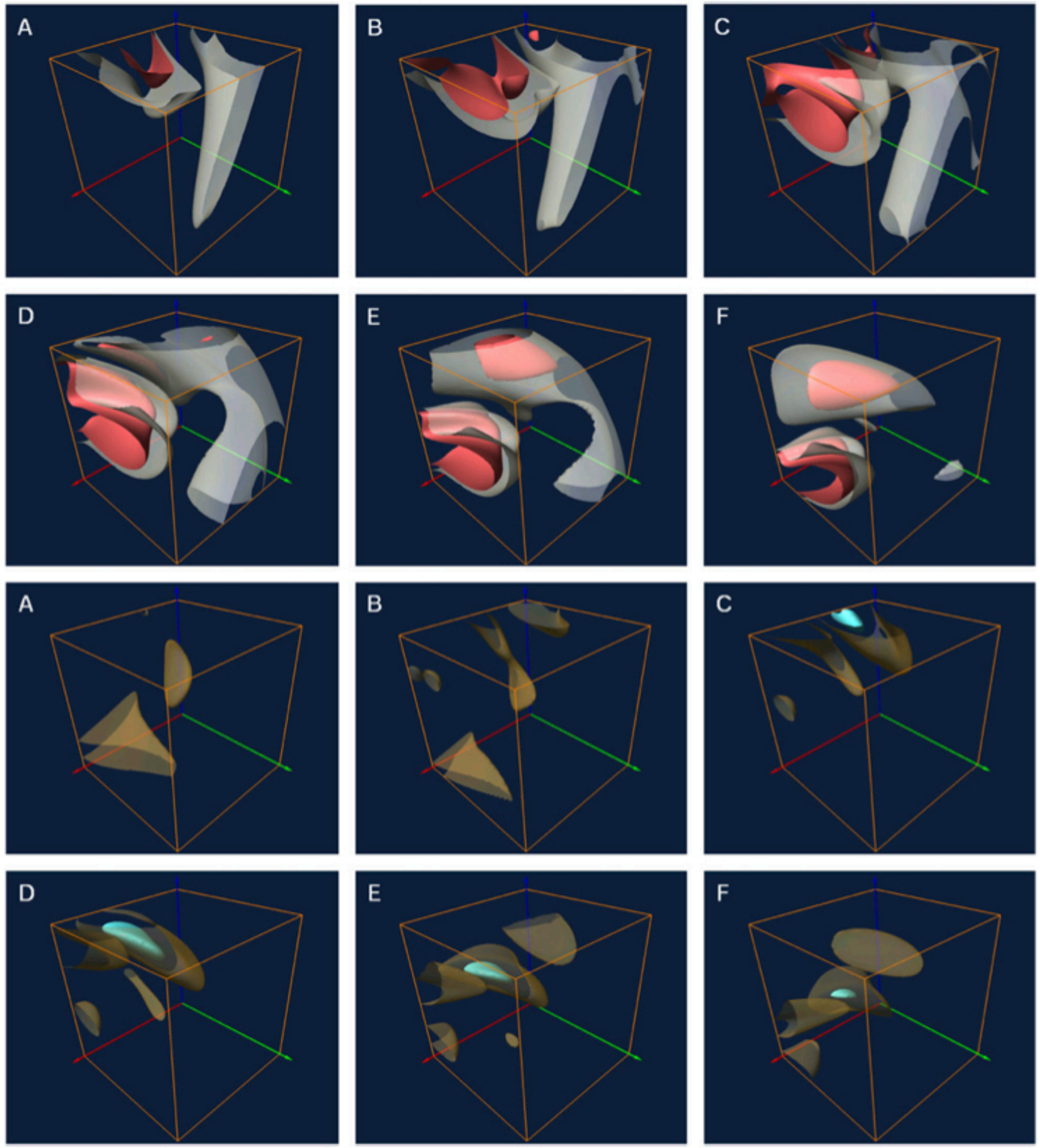
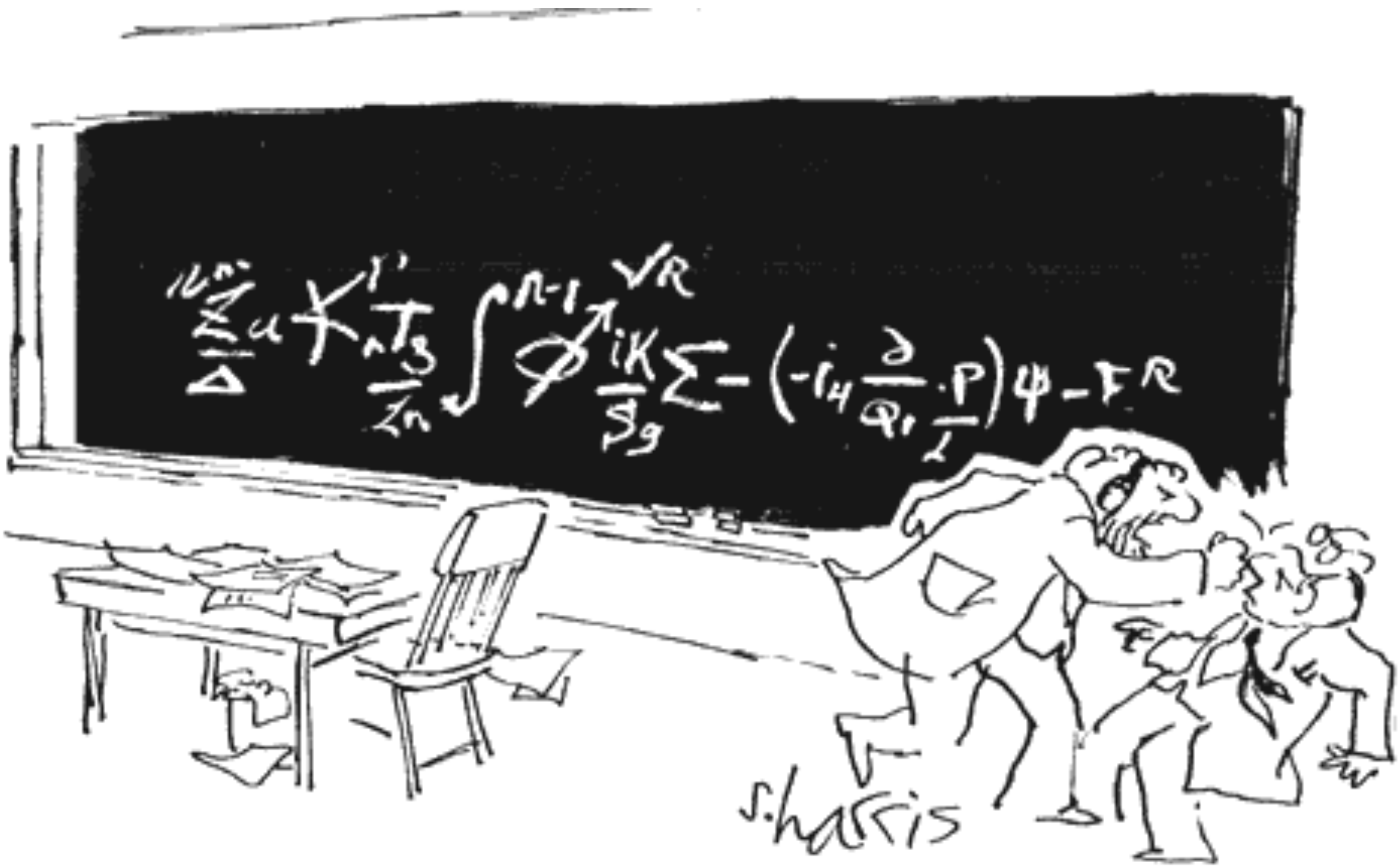


Fig. 5. (Color online.) Structures in an extreme enstrophy growth event. Upper six panels: Isosurfaces of Ω (red: 20 $\langle\Omega\rangle$, gray: 10 $\langle\Omega\rangle$). Lower six panels: Isosurfaces plots of ϵ (yellow: 2 $\langle\epsilon\rangle$, cyan: 20 $\langle\epsilon\rangle$). Panels (A) for $t/\tau_\eta = 37.44$, (B) for $t/\tau_\eta = 37.8$, (C) for $t/\tau_\eta = 38.16$ correspond with time interval I in Fig. 4; (D) for $t/\tau_\eta = 38.52$, (E) for $t/\tau_\eta = 38.88$ and (F) for $t/\tau_\eta = 39.24$ with time interval II.

AND THAT'S THAT …

THANKS FOR YOUR ATTENTION!



"You want proof? I'll give you proof!"